Intro video: Section 2.7 & 2.8
The derivative as a function

Math F251X: Calculus I

Example: A ball is thrown in the air and its height after t seconds is given by $s(t) = 40t - 16t^2$. What is its velocity after 2 s.? 1 s? ½ s? When is the velocity equal to zero?

Velocity at $t=2$ $= s'(2) = \lim_{h\to 0} \dfrac{s(2+h)-s(2)}{2+h-2}$

$= \lim_{h\to 0} \frac{1}{h}\left[\left(40(2+h) - 16(2+h)^2\right) - \left(40(2) - 16(2)^2\right)\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[40(2) + 40h - 16(4+4h+h^2) - 40(2) + 16(2)^2\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{40(2)} + 40h - \cancel{16\cdot 4} - 16\cdot 4h - 16h^2 - \cancel{40(2)} + \cancel{16(4)}\right]$

$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[40\cancel{h} - 16\cdot 4\cancel{h} - 16h^{\cancel{2}}\right] = \lim_{h\to 0}\left[40 - 16\cdot 4 - 16h\right] = 40 - 16\cdot 4$

$= -24$

Velocity at $\boxed{t = 1/2}$ $= s'(1/2) = \lim_{h\to 0} \dfrac{s(1/2+h) - s(1/2)}{1/2 + h - 1/2}$

$= \lim_{h\to 0} \frac{1}{h}\left[\left(40(\frac{1}{2}+h) - 16(\frac{1}{2}+h)^2\right) - \left(40(1/2) - 16(1/2)^2\right)\right]$ . . .

Didn't I just do this?

The derivative as a function:

Given $f(x)$, we define

$$f'(x) \underset{\text{is defined to be}}{:=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For each x, the output of the function $f'(x)$ is the derivative of $f(x)$ = instantaneous rate of change at x
= slope of tangent line at $(x, f(x))$.

Example: $s(t) = 40t - 16t^2$. What is $s'(t)$? $\boxed{s'(t) = 40 - 32t}$

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\left(40(t+h) - 16(t+h)^2\right) - \left(40t - 16t^2\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[40t + 40h - 16\left(t^2 + 2th + h^2\right) - 40t + 16t^2\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cancel{40t} + 40h - \cancel{16t^2} - 32th - 16h^2 - \cancel{40t} + \cancel{16t^2}\right] = \lim_{h \to 0} \frac{1}{\cancel{h}}\left(40\cancel{h} - 32t\cancel{h} - 16h^{\cancel{2}}\right)$$

$$= \lim_{h \to 0} 40 - 32t - 16h$$

$$= 40 - 32t + 0$$

Recap: $s(t) = 40t - 16t^2$

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \dots = 40 - 32t$$

What is the velocity after 2 seconds?

$s'(2) = 40 - 32(2) = 40 - 64 = -24$

after one second?

$s'(1) = 40 - 32 = 8$

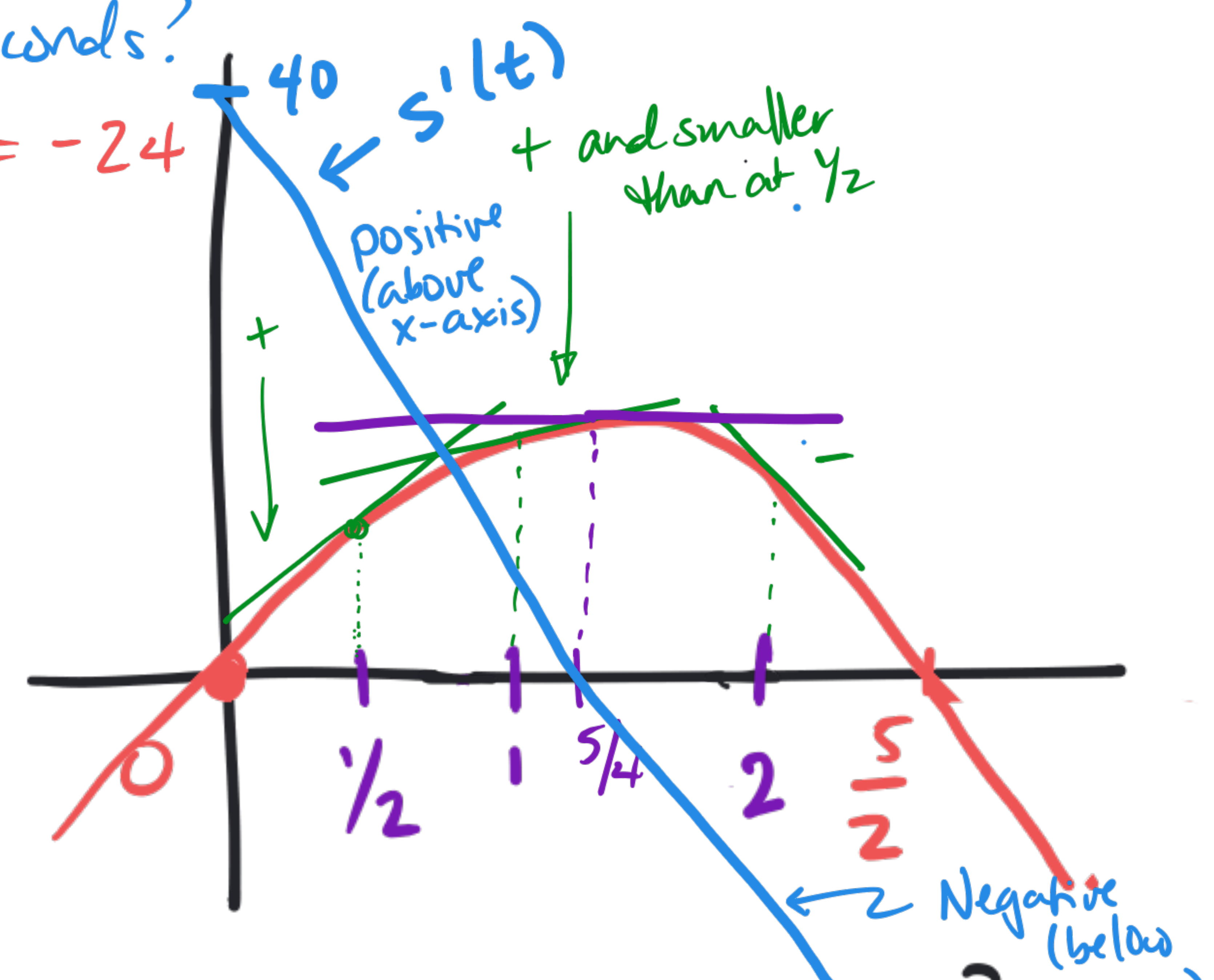
after ½ second?

$$s'(1/2) = 40 - 32(1/2) = 40 - 16 = 24$$

Where is the derivative equal to zero?

$s'(t) = 0 \Rightarrow 40 - 32t = 0$

$\Rightarrow t = \frac{-40}{-32} = \frac{5}{4}$



$$S(t) = 40t - 16t^2$$
$$= 8t(5 - 2t)$$

Example:

$f(x) = \sqrt{x} + 2x$. What is $f'(x)$, as a function?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} + 2(x+h) - \sqrt{x} - 2x}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} + \lim_{h \to 0} \frac{2(x+h) - 2x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) + \lim_{h \to 0} \frac{2x + 2h - 2x}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} + \lim_{h \to 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{(\sqrt{x+h} + \sqrt{x})\cancel{h}} + \lim_{h \to 0} 2$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} + \lim_{h \to 0} 2$$

$$= \frac{1}{2\sqrt{x}} + 2$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}} + 2}$$